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有限尺寸压电陶瓷圆板的耦合振动

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摘 要

本文从压电陶瓷圆板的二维运动方程出发，在进行分离变量后可得到仅依赖于轴向坐标 z 的方程. 然后根据边界条件用伴随法求解，从而得到了依赖于径向扩张基频的振动频谱. 我们再从厚度振动的基频出发，求其径向扩张振动的频谱. 由计算结果看出，压电陶瓷圆板的径向和轴向振动频谱不但与其压电、介电、弹性常数有关，而且它们二者之间也相互影响，即是所谓的"耦合"关系. 为了验证理论的正确性，作者计算了一个实例，计算结果与实测虽符合得不很好，但比一维理论确好得多，而所取材料参数的误差也是引起误差的一个重要因素.

一、引 言

当压电陶瓷圆板的径向与厚度的尺寸接近时(一般指其差在十倍以内)，我们称它是有限尺寸的. 对有限尺寸的压电陶瓷圆板，厚度振动常与径向振动耦合在一起形成所谓的耦合振动，其模式是非常复杂的. 许多作者试图解决这一问题并作了种种的努力[1-4]. 但迄今为止所提的方法都是近似的，还没有一个比较完整的处理方法. 本文拟从压电陶瓷圆板的二维运动方程出发，用数学上的伴随法找出其径向扩张振动与厚度振动的耦合关系，最后得到有限尺寸压电陶瓷圆板的耦合振动频谱.

二、运 动 方 程

设有一压电陶瓷圆板，半径为 R，厚度为 l，两端面涂满电极，沿厚度方向极化，工作时信号加在厚度方向，如图1所示. 由于我们讨论的是圆板，故用圆柱坐标较为方便. 在圆柱坐标中，压电陶瓷圆板的压电方程为:

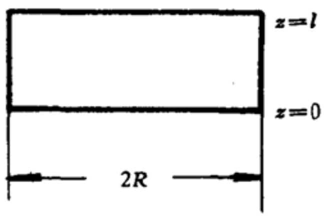

图1 压电陶瓷圆板
The piezo-ceramic disk

$$
\left.\begin{aligned}
T_{rr} &= c_{11}^{E}\frac{\partial u}{\partial r} + c_{12}^{E}\frac{u}{r} + c_{13}^{E}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \varphi}{\partial z}\\
T_{\theta\theta} &= c_{12}^{E}\frac{\partial u}{\partial r} + c_{11}^{E}\frac{u}{r} + c_{13}^{E}\frac{\partial w}{\partial z} + e_{31}\frac{\partial \varphi}{\partial z}\\
T_{zz} &= c_{13}^{E}\frac{\partial u}{\partial r} + c_{13}^{E}\frac{u}{r} + c_{33}^{E}\frac{\partial w}{\partial z} + e_{33}\frac{\partial \varphi}{\partial z}\\
T_{rz} &= c_{44}^{E}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + e_{15}\frac{\partial \varphi}{\partial r}\\
D_{r} &= e_{15}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) - \epsilon_{11}^{s}\frac{\partial \varphi}{\partial r}\\
D_{z} &= e_{31}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) + e_{33}\frac{\partial w}{\partial z} - \epsilon_{33}^{s}\frac{\partial \varphi}{\partial z}
\end{aligned}\right\} \tag{1}
$$

式中：T_{ij} 为各应力分量，D_i 为各电位移分量，u 为沿径向位移，w 为沿轴向位移，c_{ij}^{E} 为各弹性常数，e_{ij} 为各压电常数，ϵ_{ii}^{s} 为各介电常数，φ 为电势．由于圆板是轴对称的，故 $T_{r\theta}$，$T_{\theta z}$，D_θ 诸量就不存在了．

压电陶瓷圆板的运动方程为：

$$
\left.\begin{aligned}
\rho\ddot{u} &= \frac{\partial}{\partial r}T_{rr} + \frac{\partial}{\partial z}T_{rz} + \frac{T_{rr}-T_{\theta\theta}}{r}\\
\rho\ddot{w} &= \frac{\partial}{\partial r}T_{rz} + \frac{T_{rz}}{r} + \frac{\partial}{\partial z}T_{zz}\\
&\frac{\partial}{\partial r}D_r + \frac{D_r}{r} + \frac{\partial}{\partial z}D_z = 0
\end{aligned}\right\} \tag{2}
$$

式中：ρ 为质量密度，$\ddot{u}$ 和 $\ddot{w}$ 为沿径向和轴向的加速度．这里，沿 θ 方向的运动方程没有了，而式中的第三个方程代表电荷守恒定律．

将(1)式中诸量代入(2)式并假定运动是简谐的，其角频率为 ω 则有：$\ddot{u} = -\omega^2 u$ 和 $\ddot{w} = -\omega^2 w$，于是(2)式中的三个方程就变为：

$$
\left.\begin{aligned}
&c_{44}^{E}\left(\frac{\partial^2 u}{\partial r\partial z} + \frac{\partial^2 w}{\partial r^2}\right) + e_{15}\frac{\partial^2\varphi}{\partial r^2} + c_{44}^{E}\frac{1}{r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) + e_{15}\frac{1}{r}\frac{\partial\varphi}{\partial r}\\
&\quad + c_{13}^{E}\left(\frac{\partial^2 u}{\partial z\partial r} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + c_{33}^{E}\frac{\partial^2 w}{\partial z^2} + e_{33}\frac{\partial^2\varphi}{\partial z^2}\\
&\quad + \rho\omega^2 w = 0\\
&c_{11}^{E}\frac{\partial^2 u}{\partial r^2} + c_{12}^{E}\frac{1}{r}\frac{\partial u}{\partial r} - c_{12}^{E}\frac{u}{r^2} + c_{13}^{E}\frac{\partial^2 w}{\partial r\partial z} + e_{31}\frac{\partial^2\varphi}{\partial r\partial z} + c_{44}^{E}\left(\frac{\partial^2 u}{\partial z^2}\right.\\
&\quad \left. + \frac{\partial^2 w}{\partial z\partial r}\right) + e_{15}\frac{\partial^2\varphi}{\partial z\partial r} + \frac{1}{r}(c_{11}^{E} - c_{12}^{E})\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\\
&\quad + \rho\omega^2 u = 0\\
&e_{15}\left(\frac{\partial^2 u}{\partial r\partial z} + \frac{\partial^2 w}{\partial r^2}\right) - \epsilon_{11}^{s}\frac{\partial^2\varphi}{\partial r^2} + e_{15}\frac{1}{r}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) - \epsilon_{11}^{s}\frac{1}{r}\frac{\partial\varphi}{\partial r}\\
&\quad + e_{31}\left(\frac{\partial^2 u}{\partial z\partial r} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + e_{33}\frac{\partial^2 w}{\partial z^2} - \epsilon_{33}^{s}\frac{\partial^2\varphi}{\partial z^2} = 0
\end{aligned}\right\} \tag{3}
$$

我们用分离变量法解上面的三个偏微分方程，设：

$$\left.\begin{aligned} w &= F(r)f(z) \\ \varphi &= G(r)g(z) \\ u &= H(r)h(z) \end{aligned}\right\} \tag{4}$$

代入(3)式得：

$$\begin{aligned} & c_{44}^{E}\left(\frac{dH}{dr}\frac{dh}{dz}+\frac{d^2F}{dr^2}\cdot f\right)+e_{15}\frac{d^2G}{dr^2}\cdot g+c_{44}^{E}\frac{1}{r}\left(H\frac{dh}{dz}+\frac{dF}{dr}\cdot f\right) \\ &\quad + e_{15}\frac{1}{r}\frac{dG}{dr}\cdot g+c_{13}^{E}\frac{dH}{dr}\frac{dh}{dz}+c_{13}^{E}\frac{1}{r}H\frac{dh}{dz}+c_{33}^{E}F \\ &\quad \cdot\frac{d^2f}{dz^2}+e_{33}G\frac{d^2g}{dz^2}+\rho\omega^2F\cdot f=0 \\ & c_{11}^{E}\frac{d^2H}{dr^2}\cdot h+c_{12}^{E}\frac{1}{r}\frac{dH}{dr}\cdot h-c_{12}^{E}\frac{1}{r^2}H\cdot h+c_{13}^{E}\frac{dF}{dr}\cdot\frac{df}{dz} \\ &\quad + e_{31}\frac{dG}{dr}\cdot\frac{dg}{dz}+c_{44}^{E}\left(H\cdot\frac{d^2h}{dz^2}+\frac{dF}{dr}\cdot\frac{df}{dz}\right) \\ &\quad + e_{15}\frac{dG}{dr}\cdot\frac{dg}{dz}+(c_{11}^{E}-c_{12}^{E})\frac{1}{r}\left(\frac{dH}{dr}h-\frac{1}{r}H\cdot h\right) \\ &\quad + \rho\omega^2H\cdot h=0 \\ & e_{15}\left(\frac{dH}{dr}\cdot\frac{dh}{dz}+\frac{d^2F}{dr^2}\cdot f\right)-\epsilon_{11}^{s}\frac{d^2G}{dr^2}\cdot g+e_{15}\frac{1}{r}\left(H\cdot\frac{dh}{dz}+\frac{dF}{dr}\cdot f\right) \\ &\quad -\epsilon_{11}^{s}\frac{1}{r}\frac{dG}{dr}\cdot g+e_{31}\left(\frac{dH}{dr}\cdot\frac{dh}{dz}+\frac{1}{r}H\frac{dh}{dz}\right) \\ &\quad + e_{33}F\cdot\frac{d^2f}{dz^2}-\epsilon_{33}^{s}G\cdot\frac{d^2g}{dz^2}=0 \end{aligned} \tag{5}$$

可以证明[5]；$F=G$，$H=dF/dr$，则上式可分离变量，且 F 是零级贝塞尔方程的解：$F=J_0(k_r r)$。式中 k_r 原为分离常数，现成为径向波数．分离出来的三个方程为：

$$\left.\begin{aligned} & c_{33}^{E}\frac{d^2f}{dz^2}+e_{33}\frac{d^2g}{dz^2}+\rho\omega^2f-k_r^2\left[(c_{13}^{E}+c_{44}^{E})\frac{dh}{dz}+c_{44}^{E}f+e_{15}g\right]=0 \\ & (c_{13}^{E}+c_{44}^{E})\frac{df}{dz}+(e_{31}+e_{15})\frac{dg}{dz}+c_{44}^{E}\frac{d^2h}{dz^2}+(\rho\omega^2-c_{11}^{E}k_r^2)h=0 \\ & e_{33}\frac{d^2f}{dz^2}-\epsilon_{33}^{s}\frac{d^2g}{dz^2}-k_r^2\left[(e_{15}+e_{31})\frac{dh}{dz}+e_{15}f-\epsilon_{11}^{s}g\right]=0 \end{aligned}\right\} \tag{6}$$

三、伴随法解运动方程

为了用伴随法解(6)式中的三个联立方程，我们首先把它们化成标准形式．为此，从(6)式中解出 $\frac{\partial^2 f}{\partial z^2}$，$\frac{\partial^2 g}{\partial z^2}$ 和 $\frac{\partial^2 h}{\partial z^2}$ 有：

$$\left.\begin{aligned}\frac{\partial^2 f}{\partial z^2} &= A_{12}f + A_{14}g + A_{15}\frac{\partial h}{\partial z}\\ \frac{\partial^2 g}{\partial z^2} &= A_{32}f + A_{34}g + A_{35}\frac{\partial h}{\partial z}\\ \frac{\partial^2 h}{\partial z^2} &= A_{51}\frac{\partial f}{\partial z} + A_{53}\frac{\partial g}{\partial z} + A_{56}h\end{aligned}\right\} \tag{7}$$

式中:

$$A_{12} = \frac{(c_{44}^E\varepsilon_{33}^s + e_{15}e_{33})k_r^2 - \varepsilon_{33}^s\rho\omega^2}{e_{33}^2 + c_{33}^E\varepsilon_{33}^s}$$

$$A_{14} = \frac{(e_{15}\varepsilon_{33}^s - e_{33}\varepsilon_{11}^s)k_r^2}{e_{33}^2 + c_{33}^E\varepsilon_{33}^s}$$

$$A_{15} = \frac{[\varepsilon_{33}^s(c_{13}^E + c_{44}^E) + e_{33}(e_{15} + e_{31})]k_r^2}{e_{33}^2 + c_{33}^E\varepsilon_{33}^s}$$

$$A_{32} = \frac{(c_{44}^E e_{33} - c_{33}^E e_{15})k_r^2 - e_{33}\rho\omega^2}{e_{33}^2 + c_{33}^E\varepsilon_{33}^s}$$

$$A_{34} = \frac{(c_{33}^E\varepsilon_{11}^s + e_{33}e_{15})k_r^2}{e_{33}^2 + c_{33}^E\varepsilon_{33}^s}$$

$$A_{35} = \frac{[e_{33}(c_{13}^E + c_{44}^E) - c_{33}^E(e_{15} + e_{31})]k_r^2}{e_{33}^2 + c_{33}^E\varepsilon_{33}^s}$$

$$A_{51} = -\frac{c_{44}^E + c_{13}^E}{c_{44}^E}$$

$$A_{53} = -\frac{e_{31} + e_{15}}{c_{44}^E}$$

$$A_{56} = \frac{c_{11}^E k_r^2 - \rho\omega^2}{c_{44}^E}.$$

令 $\frac{\partial f}{\partial z} = f_1$, $\frac{\partial g}{\partial z} = g_1$, $\frac{\partial h}{\partial z} = h_1$, 则 $\frac{\partial^2 f}{\partial z^2} = \frac{\partial f_1}{\partial z}$, $\frac{\partial^2 g}{\partial z^2} = \frac{\partial g_1}{\partial z}$, $\frac{\partial^2 h}{\partial z^2} = \frac{\partial h_1}{\partial z}$,代入(7)式有:

$$\left.\begin{aligned}\frac{\partial f_1}{\partial z} &= A_{12}f + A_{14}g + A_{15}h_1\\ \frac{\partial f}{\partial z} &= f_1\\ \frac{\partial g_1}{\partial z} &= A_{32}f + A_{34}g + A_{35}h_1\\ \frac{\partial g}{\partial z} &= g_1\\ \frac{\partial h_1}{\partial z} &= A_{51}f_1 + A_{53}g_1 + A_{56}h\\ \frac{\partial h}{\partial z} &= h_1\end{aligned}\right\} \tag{8}$$

将上式写成矩阵的形式有:

$$\begin{pmatrix} \frac{\partial f_1}{\partial z} \\ \frac{\partial f}{\partial z} \\ \frac{\partial g_1}{\partial z} \\ \frac{\partial g}{\partial z} \\ \frac{\partial h_1}{\partial z} \\ \frac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & A_{12} & 0 & A_{14} & A_{15} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{32} & 0 & A_{34} & A_{35} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ A_{51} & 0 & A_{53} & 0 & 0 & A_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f \\ g_1 \\ g \\ h_1 \\ h \end{pmatrix} \tag{9}$$

(9) 式的伴随方程为[6]:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = - \begin{pmatrix} 0 & 1 & 0 & 0 & A_{51} & 0 \\ A_{12} & 0 & A_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & A_{53} & 0 \\ A_{14} & 0 & A_{34} & 0 & 0 & 0 \\ A_{15} & 0 & A_{35} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & A_{56} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \tag{10}$$

将 $x_1\, x_2\, x_3\, x_4\, x_5\, x_6$ 依次乘(9)中诸式，而将 $f_1\, f\, g_1\, g\, h_1\, h$ 依次乘(10)中诸式，并将所得两式按左右端相加，可以看出右端各项相加后变为零，左端相加后成为全微分，故有：

$$\frac{d}{dz}(x_1f_1 + x_2f + x_3g_1 + x_4g + x_5h_1 + x_6h) = 0$$

将此式在区间 $0<z<l$ 上积分后得：

$$\begin{aligned} & x_1(l)f_1(l) + x_2(l)f(l) + x_3(l)g_1(l) + x_4(l)g(l) + x_5(l)h_1(l) + x_6(l)h(l) \\ & \quad - x_1(0)f_1(0) - x_2(0)f(0) - x_3(0)g_1(0) - x_4(0)g(0) - x_5(0)h_1(0) \\ & \quad - x_6(0)h(0) = 0 \end{aligned} \tag{11}$$

式中括号内的 l 或 0 分别表示各该量在 $z=l$ 处或 $z=0$ 处的值．所有的 $f_1\, f\, g_1\, g\, h_1\, h$ 暂时还是未知数，而 $x_1x_2\cdots$ 则可从(10)式根据边界条件求出．下面我们就来求这些函数．

首先来看压电陶瓷圆板的边界条件．在 $z=0$ 处有：$T_{zz}=T_{rz}=0$ 和 $\varphi=0$，注意到(1)式有：

在 $z=0$ 处：

$$c_{33}^E f_1(0) + e_{33}g_1(0) - k_r^2 c_{13}^E h(0) = 0 \tag{12}$$

$$c_{44}^E h_1(0) + c_{44}^E f(0) + e_{15}g(0) = 0 \tag{13}$$

$$g(0) = 0 \tag{14}$$

同样，在 $z=l$ 处有 $T_{zz}=T_{rz}=0$ 和 $\varphi=0$，仿照上面的运算得：

在 $z=l$ 处：

$$c_{33}^E f_1(l) + e_{33}g_1(l) - k_r^2 c_{13}^E h(l) = 0 \tag{15}$$

$$c_{44}^E h_1(l) + c_{44}^E f(l) + e_{15} g(0) = 0 \tag{16}$$

$$g(l) = 0 \tag{17}$$

因此,我们分别用如下的终端条件:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} c_{33}^E \\ 0 \\ e_{33} \\ 0 \\ 0 \\ -k_r^2 c_{13}^E \end{pmatrix}^{(1)}, \begin{pmatrix} 0 \\ c_{44}^E \\ 0 \\ e_{15} \\ c_{44}^E \\ 0 \end{pmatrix}^{(2)}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}^{(3)} \tag{18}$$

依次对(10)式进行反积分,解出:

$$\left.\begin{matrix} x_1^{(1)}(z)x_2^{(1)}(z)x_3^{(1)}(z)x_4^{(1)}(z)x_5^{(1)}(z)x_6^{(1)}(z) \\ x_1^{(2)}(z)x_2^{(2)}(z)x_3^{(2)}(z)x_4^{(2)}(z)x_5^{(2)}(z)x_6^{(2)}(z) \\ x_1^{(3)}(z)x_2^{(3)}(z)x_3^{(3)}(z)x_4^{(3)}(z)x_5^{(3)}(z)x_6^{(3)}(z) \end{matrix}\right\} \tag{19}$$

将(19)式中诸函数分别代入(11)式,并注意到(15)—(17)式则有:

$$x_1^{(1)}(0)f_1(0) + x_2^{(1)}(0)f(0) + x_3^{(1)}(0)g_1(0) + x_4^{(1)}(0)g(0) + x_5^{(1)}(0)h_1(0) + x_6^{(1)}(0)h(0) = 0 \tag{20}$$

$$x_1^{(2)}(0)f_1(0) + x_2^{(2)}(0)f(0) + x_3^{(2)}(0)g_1(0) + x_4^{(2)}(0)g(0) + x_5^{(2)}(0)h_1(0) + x_6^{(2)}(0)h(0) = 0 \tag{21}$$

$$x_1^{(3)}(0)f_1(0) + x_2^{(3)}(0)f(0) + x_3^{(3)}(0)g_1(0) + x_4^{(3)}(0)g(0) + x_5^{(3)}(0)h_1(0) + x_6^{(3)}(0)h(0) = 0 \tag{22}$$

将方程(12)(13)(14)(20)(21)(22)联立起来并写成矩阵的形式有:

$$\begin{pmatrix} c_{33}^E & 0 & e_{33} & 0 & 0 & -k_r^2 c_{13}^E \\ 0 & c_{44}^E & 0 & e_{15} & c_{44}^E & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ x_1^{(1)}(0) & x_2^{(1)}(0) & x_3^{(1)}(0) & x_4^{(1)}(0) & x_5^{(1)}(0) & x_6^{(1)}(0) \\ x_1^{(2)}(0) & x_2^{(2)}(0) & x_3^{(2)}(0) & x_4^{(2)}(0) & x_5^{(2)}(0) & x_6^{(2)}(0) \\ x_1^{(3)}(0) & x_2^{(3)}(0) & x_3^{(3)}(0) & x_4^{(3)}(0) & x_5^{(3)}(0) & x_6^{(3)}(0) \end{pmatrix} \begin{pmatrix} f_1(0) \\ f(0) \\ g_1(0) \\ g(0) \\ h_1(0) \\ h(0) \end{pmatrix} = 0 \tag{23}$$

可以看出,(23)式是齐次的,假如它们有解,其系数行列式必须等于零,这就使我们得到压电陶瓷圆板厚度振动的频率方程. 解出频率方程就得到压电陶瓷圆板厚度振动的频谱. 把每一个频率代入(23)式,就可解出对应于该频率的 $f_1(0)$ $f(0)$ $g_1(0)$ $g(0)$ $h_1(0)$ $h(0)$ 诸函数. 当然,这六个函数中有一个是任意常数,它与接到压电陶瓷的驱动源有关. 以这六个函数作为初条件代入(9)式,就可解出 $f_1(z)$ $f(z)$ $g_1(z)$ $g(z)$ $h_1(z)$ $h(z)$,也就是说得到了对应于该频率的振动模式.

四、数值计算例

作者用此法计算了一个实例,所取压电陶瓷的直径为 $\phi 31.5$mm,厚度为 5.5mm,有关参数取为: $f = 7.627$, $\varepsilon_{33}^s = 788$, $\varepsilon_{11}^s = 397$, $c_{11}^E = 12.54 \times 10^{10}(\mathrm{N/m^2})$, $c_{12}^E = 7.16 \times 10^{10}$,

$c_{13}^E = 6.85 \times 10^{10}$，$c_{33}^E = 14.1 \times 10^{10}$，$c_{44}^E = 2.07 \times 10^{10}$，$e_{15} = 10.23(\mathrm{N/m^2})$，$e_{33} = 16.84$，$e_{31} = -8.40$，$\sigma = 0.31$.

作为零级近似，我们先以薄圆片径向扩张基频的波数 $k_{r_0} = 2.05/1.575 = 1.3016/\mathrm{cm}$，代入(7)中 A_{12}—A_{56} 诸系数，然后由(23)的系数行列式为零的条件解出 ω 的两个最低根 ω_0 和 ω_1，即厚度振动的基频及一次谐频．再将 ω_0 和 ω_1 分别代入(23)，算出对应的 $f_1(0)$—$h(0)$．再将这些初函数代入(9)式，算出对应的 $f_1(z)$—$h(z)$ 诸函数，按照文献[5]的方法对 k_{r_0} 进行修正得 $\Delta k_{r_{01}}$ 和 $\Delta k_{r_{02}}$，然后再以 $k_{r_0} + \Delta k_{r_{01}}$ 和 $k_{r_0} + \Delta k_{r_{02}}$ 作为一级近似的 k_r 重复上述的计算，最后得一对 ω_0 和一对 ω_1．再以两个 ω_0 分别代入(23)，而将 k_r 看作变量每次都算出 k_r 的五个根．当然我们也可以两个 ω_1 代入(23)作同样的计算，但算出的数据与实测差得更大．所以说，实际上影响径向扩张振动频率的是 ω_0 而不是 ω_1.

最后，我们将实测的值，用本法计算的值，以及用一维近似计算的值加以对照列于表 1．由表 1 可以看出，与一维近似计算比较，本法的计算更接近于实测．但也存在一些问题，例如计算值与实测虽然大致能对应，但误差还较大．另外，实测的频谱中还有一个频率即297.431kHz，没有对应的计算值

表 1　计算值、实测值对照表

Comparison of experimental results with theoretical

本法计算值 Theory (this paper)	实 测 值 Measurements	一维计算值 One-dimensional theory
$f_{r_0}^{(1)} = 71.426$(kHz) $f_{r_0}^{(2)} = 71.652$	71.281	$f_{r_0} = 72.665$
$f_{r_1}^{(1)} = 178.344$ $f_{r_1}^{(2)} = 183.665$	176.469	$f_{r_1} = 190.600$
$f_{r_2}^{(1)} = 258.715$ $f_{r_2}^{(2)} = 271.364$	244.908 263.905	$f_{r_2} = 297.054$
——	297.431	——
$f_{r_3}^{(1)} = 375.362$ $f_{r_3}^{(2)} = 393.236$	366.834 388.305	$f_{r_3} = 415.718$
$f_{r_4}^{(1)} = 434.455$ $f_{r_4}^{(2)} = 461.772$	401.194 457.903	$f_{r_4} = 528.129$
$f_{z_{01}} = 332.854$ $f_{z_{02}} = 345.910$	327.786 343.402	$f_{z_0} = 347.880$
$f_{z_{11}} = 1244.954$ $f_{z_{12}} = 1285.113$	1237.574	$f_{z_1} = 1259.317$

五、结 语

本文是将计算数学中的伴随法用于解耦合振动问题的一个尝试．径向与厚度谐振频率之间的耦合关系明确地表示在(23)中的频率方程中．所得结果与实测基本相符，造成误差的原因还包括所取材料参数可能与实际材料的性能有出入．本文的数值计算工作是上海电器成套研究所电子计算机室吴福民同志进行的，在此表示谢意．

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THE FREQUENCY SPECTRUM OF COUPLED-VIBRATIONS OF FINITE CIRCULAR PIEZOELECTRIC CERAMIC DISKS

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ABSTRACT

This paper based on two dimensional motion equations of circular piezoceramic disks, and separate the variables we can attain the equation set which depends on axial variable z only, then according to the boundary conditions solved this equation set with adjoint method to obtain the vibrational frequency spectrum which is relative to the radial expansive frequencies. Conversely from the basic frequency of thickness vibrations we could calculate the spectrum of radial expansive vibrations. From the results of calculation by this method, we can see that the radial and axial vibration spectrum of circular piezoceramic disks are influenced by each other as well as dependence to their dielectric, piezoelectric and elastic constans. That is so called "Coupling" relations.

In order to examine the correctness of this method, the author calculated an example. Although the results of calculation is not very coincide with measurements, but is better than that of onedimensional theory. Here the errors in material parameters are important factor to cause mistakes.